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NUMERICAL APPROXIMATION OF A CAUCHY
PROBLEM FOR A PARABOLIC PARTIAL
DIFFERENTIAL EQUATION

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ABSTRACT

A procedure for the numerical approximation of the Cauchy problem for
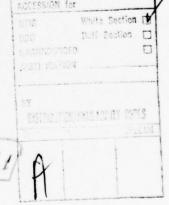the following linear parabolic partial differential equation is defined:

$$u_t - (p(x)u_x)_x + q(x)u = 0 , \quad 0 < x < 1 , \quad 0 < t \leq T; \quad u(0,t) = f_1(t),$$

$$0 < t \leq T; \quad u(1,t) = f_2(t), \quad 0 < t \leq T; \quad p(0)u_x(0,t) = g(t),$$

$$0 < t_0 \leq t \leq T; \quad |u(x,t)| \leq M , \quad 0 \leq x \leq 1, \quad 0 \leq t \leq T .$$

The procedure involves Galerkin type numerical methods for related parabolic
initial boundary-value problems and a linear programming problem. Explicit
a priori error estimates are presented for the entire discrete procedure
when the data f_1, f_2, and g are known only approximately.

AMS(MOS) Subject Classifications: 65M30, 65N30

Key Words: Cauchy problem, error estimates, improperly posed problem.

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SIGNIFICANCE AND EXPLANATION

In many physical problems in heat conduction, it is impossible to obtain an initial temperature distribution within a material. In many of these cases, in order to obtain approximations of the temperature within the body, one must rely entirely upon data which can be measured at the boundary. An additional problem is that these boundary data are only accurate to within some prescribed measurement errors.

The purpose of this paper is to define a procedure for numerically approximating the solution of one such heat flow problem and to present explicit error estimates for the numerical procedure. A priori error estimates are presented when the data are known only approximately.

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NUMERICAL APPROXIMATION OF A CAUCHY PROBLEM
FOR A PARABOLIC PARTIAL DIFFERENTIAL EQUATION

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1. Introduction.

Consider the numerical approximation of the solution of the following Cauchy problem for a linear parabolic partial differential equation.

Problem (P): Find a function $u = u(x,t)$ satisfying

a) $\dfrac{\partial u}{\partial t} - \dfrac{\partial}{\partial x}\left(p(x)\,\dfrac{\partial u}{\partial x}\right) + q(x)u = 0$, $0 < x < 1$, $0 < t \le T$,

b) $u(0,t) = f_1(t)$ $0 < t \le T$,

(1.1)   c) $u(1,t) = f_2(t)$, $0 < t \le T$,

d) $p(0)\,\dfrac{\partial u}{\partial x}\,(0,t) = g(t)$, $0 < t_0 \le t \le T$,

e) $|u(x,t)| \le M$, $0 \le x \le 1$, $0 \le t \le T$,

where the data f_1, f_2, and g are known only approximately as f_1^*, f_2^*, and g^* such that

a) $\|f_1 - f_1^*\|_{[0,T]} \le \varepsilon_0$,

(1.2)   b) $\|f_2 - f_2^*\|_{[0,T]} \le \varepsilon_0$,

c) $\|g - g^*\|_{[t_0,T]} \le \varepsilon_0$,

with $\varepsilon_0 > 0$ and where for any function $f = f(t)$

(1.3) $\|f\|_{[a,b]} = \sup\limits_{a \le t \le b} |f(t)|$.

We assume that the following hypotheses are satisfied:

(H1) f_1, f_2, g, p, and q are such that a classical solution u to (1.1) exists.

(H2)   The functions p, p' and q are uniformly Hölder continuous in $0 \le x \le 1$ and satisfy

a) $0 < p_* \le p(x) \le p^*$,

b) $0 \le q_* \le q(x) \le q^*$,

c) $|p'(x)| \le p'^*$.

(H3) f_1 and f_2 are continuously differentiable and a constant K_1 exists such that

$$\|f_1\|_{[0,T]} + \|f_1'\|_{[0,T]} + \|f_2\|_{[0,T]} + \|f_2'\|_{[0,T]} + \|g\|_{[t_0,T]} \le K_1.$$

(H4) For the Sturm-Liouville problem with eigenvalues and corresponding normalized

   eigenfunctions, satisfying

   a) $(p\varphi_n')' - q\varphi_n + \lambda_n \varphi_n = 0$, $0 < x < 1$,

   b) $\varphi_n(0) = \varphi_n(1) = 0$,

   there exists a $d > 0$ such that

      $d = \inf_n (\lambda_{n+1} - \lambda_n)$.

(H5) The approximations (1.2) hold. In addition

   a) $\| f_1' + f_1^{*'} \|_{[0,T]} \leq \epsilon_0$,

   b) $\| f_2' - f_2^{*'} \|_{[0,T]} \leq \epsilon_0$.

(H6) If $M > 0$ is the constant given in (1.1.e), we have

   a) $\| f_1 \|_{[0,T]} \leq M$,

   b) $\| f_2 \|_{[0,T]} \leq M$,

   c) $\| f_1^* \|_{[0,T]} \leq M$,

   d) $\| f_2^* \|_{[0,T]} \leq M$.

Without the extra assumption (1.1.e), the Cauchy problem (1.1.a-d) is not well-
posed in the sense of Hadamard [3,4,5,6,7,8,13] since the solution does not depend contin-
uously upon the data. However, for the problem as stated with hypotheses (H1)-(H6)
satisfied, a continuous dependence result was obtained in [6].

Many studies of the continuous dependence of the Cauchy problem for various para-
bolic problems have appeared in the literature [3-8. 13]. Other formulations in terms of
control problems have also appeared in the control theory literature [11, 12, 14]. In [7]
Ginsberg considered numerical approximation of the Cauchy problem for the heat equation
$u_t = u_{xx}$ with $g(t) \equiv 0$ in (1.1.e) by expanding the data in Fourier series and estimating
Fourier coefficients. In [3,4] Cannon and Douglas outlined numerical procedures for various
Cauchy problems for the heat equation by reducing the problems to mathematical programming
techniques. In [5] Cannon and one of the authors presented a direct numerical method for a
slightly different Cauchy problem for the heat equation in which a Taylor series expansion
for the data is numerically approximated. In [6] a numerical scheme requiring numerical
approximation of several unknown eigenvalues and eigenfunctions was presented without
explicit error estimates for the approximations. In this paper the numerical schemes

-2-

involve only solution of linear parabolic initial boundary-value problems and a simple linear programming problem. More importantly, explicit a priori error estimates for the entire procedure are presented.

In Section 2, basic notation is presented and Problem (P) is reduced by linearity into two simple initial boundary-value problems and an optimization problem. In Section 3, Galerkin type numerical schemes are defined for the initial boundary-value problems and a linear programming problem is formulated to solve the optimization problem. Several basic lemmas needed to prove the main result are stated in Section 4. Then a priori error estimates for obtaining approximations to (1.1) with approximate data satisfying (1.2) are stated and proved. Finally proofs of two of the technical lemmas are given.

2. Preliminaries.

We shall first define some of the notations used for various norms throughout the paper. Recall that in Section 1 we used the notation that, for any function $f = f(t)$,

$$\|f\|_{[a,b]} \equiv \sup_{a \le t \le b} |f(t)|.$$

For functions $\psi = \psi(x)$ defined on $(0,1)$, we shall denote by $|\psi|_\infty$ the norm $\|\psi\|_{L^\infty(0,1)}$ and by $W^{m,\infty}$ (m a positive integer) the usual Sobolev space of functions with norm

$$|\psi|_{m,\infty} \equiv \sum_{j=0}^{m} \left| \frac{\partial^j \psi}{\partial x^j} \right|_\infty .$$

For real s, we further denote by H^s the Sobolev space $W^{s,2}$ of real-valued functions defined on $(0,1)$ and by $\|\psi\|_s$ its corresponding norm. We note that $L^2(0,1)$ will be denoted by H^0 and $\|\psi\|_{L^2(0,1)}$ by $\|\psi\|_0$. For definitions of the other spaces, we refer the reader to [10].

Also, for X a normed space with norm $\|\cdot\|_X$ and $u:[a,b] \to X$, we define

$$\|u\|_{L^2(a,b;X)}^2 \equiv \int_a^b \|u(\cdot,t)\|_X^2 \, dt$$

and

$$\|u\|_{L^\infty(a,b;X)} \equiv \sup_{a \le t \le b} \|u(\cdot,t)\|_X .$$

Finally, for convenience, we define a bilinear form

$$a(u,v) \equiv (p(\cdot)u_x, v_x) + (q(\cdot)u,v) ,$$

where $(\cdot,\cdot)$ denotes the $L^2(0,1)$ inner product.

We shall next present a reformulation of Problem (P) on which our approximation scheme will be based. We first define a function w satisfying the initial boundary value problem

(2.1)

a) $\dfrac{\partial w}{\partial t} - \dfrac{\partial}{\partial x} \left(p \dfrac{\partial w}{\partial x} \right) + qw = F(x,t), \quad 0 < x < 1, \quad 0 < t \le T$,

b) $w(0,t) = 0, \quad 0 < t \le T$,

c) $w(1,t) = 0, \quad 0 < t \le T$,

d) $w(x,0) = 0, \quad 0 < x < 1$,

where

(2.2) $\quad F(x,t) = -(1-x)f_1'(t) - xf_2'(t) + p'(x)[f_2(t) - f_1(t)] - q[(1-x)f_1(t) + xf_2(t)].$

-4-

Also, for each $\psi \in L^2(0,1)$, we define a function z^ψ satisfying the initial boundary value problem:

(2.3)

a) $\dfrac{\partial z^\psi}{\partial t} - \dfrac{\partial}{\partial x}\left(p\,\dfrac{\partial z^\psi}{\partial x}\right) + qz^\psi = 0$, $\quad 0 < x < 1$, $\quad 0 < t \le T$,

b) $z^\psi(0,t) = 0$, $\qquad\qquad\qquad 0 < t \le T$,

c) $z^\psi(1,t) = 0$, $\qquad\qquad\qquad 0 < t \le T$,

d) $z^\psi(x,0) = \psi$. $\qquad\qquad\qquad 0 < x < 1$.

Using M from (1.1.d), let

$$K \equiv \{\varphi \in C^0[0,1]:\ |\varphi|_\infty \le M \ \text{ and }\ \varphi(0) = f_1(0),\ \varphi(1) = f_2(0)\}.$$

We can now reformulate Problem (P) using (2.1)-(2.3) and linearity of the operator in (1.1) as follows:

Find φ (corresponding to $u(x,0)$ from (1.1)) with $\varphi \in K$ such that

(2.4)

$$p(0)\,\dfrac{\partial z^{\varphi-r}}{\partial x}(0,t) = g(t) - p(0)\,\dfrac{\partial w}{\partial x}(0,t) - p(0)[f_2(t) - f_1(t)]$$

$$\equiv G(t)$$

where

(2.5) $\quad r(x) \equiv (1-x)f_1(0) + xf_2(0)$.

To see that this reformulation is equivalent to Problem (P), observe that, using linearity, the function

$$u^\psi = w + z^\psi + (1-x)f_1(t) + xf_2(t)$$

satisfies

(2.6)

a) $\dfrac{\partial u^\psi}{\partial t} - \dfrac{\partial}{\partial x}\left(p\,\dfrac{\partial u^\psi}{\partial x}\right) + qu^\psi = 0$, $\quad 0 < x < 1$, $\quad 0 < t \le T$,

b) $u^\psi(0,t) = f_1(t)$, $\qquad\qquad\qquad 0 < t \le T$,

c) $u^\psi(1,t) = f_2(t)$, $\qquad\qquad\qquad 0 < t \le T$,

d) $u^\psi(x,0) = \psi + (1-x)f_1(0) + xf_2(0)$, $\qquad\quad 0 < x < 1$,

e) $p(0)u_x^\psi(0,t) = p(0)w_x(0,t) + p(0)z_x^\psi(0,t) + p(0)[f_2(t) - f_1(t)]$, $\quad t_0 \le t \le T$.

Choosing $\psi = \varphi - r$, we see that $u^{\varphi-r}$ solves problem (P) provided we can find a φ satisfying (2.4). (We note that since $|f_1(t)| \le M$ and $|f_2(t)| \le M$, $0 \le t \le T$, the maximum principle implies that the condition $|\varphi|_\infty \le M$ is equivalent to requiring that $|u(x,t)| \le M$, for $0 \le x \le 1$, $0 \le t \le T$.)

-5-

Using this notation, we can now state a form of the continuous dependence result (proved in [6]) which we shall need later in the derivation of the error estimates.

<u>Lemma 1</u>:   Under hypotheses (H1)-(H6), there exist computable constants C_1 and γ $(0 < \gamma < 1)$ such that for all ψ with $|\psi|_\infty \leq M$,

(2.7) $\|z^\psi\|_{L^\infty(t_0,T;W^{1,\infty})} \leq C_1 \|z_x^\psi(0,\cdot)\|_{[t_0,T]}^\gamma$,

where z^ψ is the solution of (2.3).

3. Description of the Numerical Approximations.

In this section we consider the problem of numerically approximating the solution of (1.1) (or equivalently, its reformulation described in Section 2) subject to the restriction (1.2). The restriction (1.2) comes from the fact that data measurement error is, in general, accurate only to within some measurement tolerance ε_0. We denote by F^* the function obtained by replacing f_1 and f_2, in (2.2), by f_1^* and f_2^* respectively. We further define w^* as the solution of (2.1) with F replaced by the F^* defined above,

$$(3.1) \qquad r^*(x) \equiv (1-x) f_1^*(0) + x f_2^*(0)$$

and

$$(3.2) \qquad G^*(t) \equiv g^*(t) - p(0) \frac{\partial w^*}{\partial x}(0,t) - p(0) [f_2^*(t) - f_1^*(t)] .$$

We note here that from (H5), (H6), (2.5), and (3.1) we have

a) $|r|_\infty \leq M$

(3.3) b) $|r^*|_\infty \leq M$

c) $\|r-r^*\|_0 \leq |r-r^*|_\infty$

$$\leq \max_{0 \leq x \leq 1} \{ (1-x)|f_1(0) - f_1^*(0)| + x|f_2(0) - f_2^*(0)| \}$$

$$\leq \varepsilon_0 .$$

Since the data is only known approximately as described above, we now define an approximation scheme based on a finite dimensional analogue of equation (2.4) with G replaced by G^* from (3.2). We first describe schemes for obtaining approximations to w and z^ψ (assuming ψ is known).

Let S_h^β denote the space of continuous piecewise polynomials of degree $\beta-1$ defined on a uniform mesh of width h on $[0,1]$ and vanishing at $x = 0$ and $x = 1$. We shall consider a family of such spaces for $0 < h < 1$. We assume each space in this family satisfies the following so-called "inverse assumptions": $u \in S_h^\beta$ implies that for some constant C_2,

(3.4)
a) $|u|_{1,\infty} \leq C_2 h^{-1} |u|_\infty$,
b) $|u|_\infty \leq C_2 h^{-1/2} \|u\|_0 .$

We also assume that our test spaces S_h^β satisfy the following approximation assumptions: if $u \in S_h^\beta$ and $I_h u$ is the interpolate of u in S_h^β, then for some constant C_3,

-7-

$$(3.5) \quad \text{a)} \quad \|u - I_h u\|_0 \le C_3 \, h^s \|u\|_s , \quad s \ge 0 ,$$

$$\text{b)} \quad |I_h u|_{1,\infty} \le C_3 \, |u|_{1,\infty} .$$

Let $k > 0$ be the step size in time, $N_T = T/k \in \mathbf{Z}$, $t^n = nk$, and $\varphi_n = \varphi(t^n)$. We shall present a Crank-Nicolson-Galerkin approximation for w, the solution of $(2.1)-(2.2)$. Define $W : \{0 = t_0, t_1, \ldots, t_{N_T} = T\} \to S_{h_1}^4$ by

$$(3.6) \quad \left(\frac{W_{n+1} - W_n}{k}, V\right) + a\left(\frac{W_{n+1} + W_n}{2}, V\right) = (F(\cdot, (n + \tfrac{1}{2})k), V)$$

for all $V \in S_{h_1}^4$ and $n = 0, 1, \ldots, N_T - 1$, with $W_0 \equiv 0$, where $a(\cdot, \cdot)$ is defined in Section 2. We similarly define W_n^* to be the analogous approximation to $w^*(nk)$ given by $(2.1)-(2.2)$ with F replaced by F^*.

The scheme defined in (3.6) is known to have a time-truncation error of the order k^2. We shall use another $O(k^2)$ time-stepping method with better stability properties but greater work estimates to approximate z, the solution of (2.3) with ψ assumed known. Define $Z^\psi: \{0 = t_0, t_1, \ldots, t_{N_T} = T\} \to S_h^2$ by

$$\left(\frac{Z_{n+\alpha}^\psi - Z_n^\psi}{\alpha k}, v\right) + a(Z_{n+\alpha}^\psi, v) = 0 ,$$

$$(3.7)$$

$$\left(\frac{Z_{n+1}^\psi - Z_n^\psi}{\alpha k}, v\right) + a(Z_{n+1}^\psi, v) = \frac{-(1-\alpha)}{\alpha} \, a(Z_{n+\alpha}^\psi, v) ,$$

with

$$(3.8) \quad (Z_0^\psi, v) = (\psi, v) ,$$

for all $v \in S_k^2$ where $\alpha = 1 - \frac{\sqrt{2}}{2}$.

We note that since each of the time stepping schemes defined above have $O(k^2)$ time-discretization error but different spacial orders of approximation, we shall use the time step k to tie the two approximations together. Thus k will be the same in each of (3.6) and (3.7). We shall then see that in order to balance the temporal and spatial discretization errors in each problem separately, we shall let $h = k$ in the definition of Z and $h_1 = k^{1/2}$ in the definition of W.

Let $N_0 = [\![t_0/k]\!] + 1$ where $[\![\tilde{r}]\!]$, for $\tilde{r} \in \mathbb{R}$, is the greatest integer less than $\tilde{r}$. Using the above definitions, we can now define an approximate problem as follows:

Problem (P_A): Find $\varphi_h \in K_h$ such that

(3.9)
$$J(\varphi_h) = \inf_{\psi_h \in K_h} J(\psi_h) ,$$

where

a) $K_h = \{\varphi_h \in S_h^2 : |\varphi_h|_\infty \leq M, \quad \varphi_h(0) = f_1^*(0), \text{ and } \varphi_h(1) = f_2^*(0)\}$,

(3.10)

b) $J(\psi_n) = \max\limits_{n=N_0,\ldots,N_T} |g^*(nk) - p(0)[f_2^*(nk) - f_1^*(nk)]$

$$- p(0) \frac{\partial}{\partial x} W_n^*(0) - p(0) \frac{\partial}{\partial x} Z_n^{\psi_h - r^*}(0)| .$$

We then take as our approxiamtion to $u(t)$ at $t = nk$ the function

(3.11)
$$U_n = W_n^* + Z_n^{\varphi_h - r^*} + (1-x)f_1^*(nk) + x f_2^*(nk) .$$

We now show how Problem (P_A) can be solved by linear programming.

Let h be such that $H = 1/h \in \mathbb{Z}$ and

(3.12)
$$\psi_h = \sum_{i=1}^{H-1} C_i \Phi_i + f_1^*(0) \Phi_0 + f_2^*(0) \Phi_H ,$$

where

(3.13)
$$\Phi_i = \begin{cases} 0 & , \quad x \leq (i-1)h, \\ x/h - (i-1), & (i-1)h \leq x \leq ih , \\ 1+i - x/h , & ih \leq x \leq (i+1)h, \\ 0 & , \quad x \geq (i+1)h . \end{cases}$$

Note that the constraint $|\psi_h|_\infty \leq M$ is equivalent to $|C_i| \leq M$, $i = 1,\ldots,H-1$ and $|f_i^*(0)| \leq M$, $i = 1,2$. Then, by linearity,

(3.14)
$$\frac{\partial}{\partial x} Z_n^{\psi_h - r^*}(0) = \sum_{i=1}^{H-1} C_i \frac{\partial}{\partial x} Z_n^{\Phi_i}(0) + \frac{\partial}{\partial x} Z_n^{f_1^*(0)\Phi_0 + f_2^*(0)\Phi_H}(0) - \frac{\partial}{\partial x} Z_n^{r^*}(0) .$$

Hence Problem (P_A) can be written as:

Find $\vec{C} = (C_1,\ldots,C_{H-1})$ minimizing λ subject to the constraints

a) $\quad -M \leq C_i \leq M$, $i = 1,\ldots,H-1$, and

(3.15) b) $\quad -\lambda \leq g^*(nk) - p(0)[f_2^*(nk) - f_1^*(nk)] - p(0) \frac{\partial}{\partial x} W_n^*(0)$

$$-p(0) [\sum_{i=1}^{H-1} C_i \frac{\partial}{\partial x} Z_n^{\Phi_i}(0) + \frac{\partial}{\partial x} Z_n^{f_1^*(0)\Phi_0 + f_2^*(0)\Phi_H}(0) - \frac{\partial}{\partial x} Z_n^{r^*}(0)]$$

$$\leq \lambda , \qquad n = N_0,\ldots,N_T .$$

-9-

4. Main Results.

In order to derive our main result, we shall need several lemmas about the regularity and approximation of the solutions of (2.1)-(2.2) and (2.3) and the stability of problem (3.6). We shall now state these lemmas. The first lemma relates the smoothness of the solution of (2.3) to its initial smoothness and can be found in [2].

<u>Lemma 2</u>: Let Z^ψ be the solution of problem (2.3). Then for $0 < t_0 \leq t$ and $s \geq 0$, there exists a constant C_4 such that

$$(4.1) \qquad \| Z^\psi(\cdot,t) \|_s \leq C_4 t_0^{-s/2} \| \psi \|_0 .$$

<u>Lemma 3:</u> (c.f. Wheeler[15]). Let w be the solution of (2.1)-(2.2) and $\{W_n\}$ be its approximation given by (3.6). If $w \in L^\infty(0,T;W^{4,\infty})$, $\frac{\partial w}{\partial t} \in L^2(0,T;H^4)$, and $\frac{\partial^3 w}{\partial t^3} \in L^2(0,T;H^0)$, then there are constants C_4 and $k_0 > 0$ such that for all $0 < k < k_0$, we have for $n = 0,\ldots,N_T$,

$$(4.2) \quad |w(\cdot,nk) - W_n|_\infty \leq C_4 \{ h_1^4 [\| w \|_{L^\infty(0,T;W^{4,\infty})} + \| \frac{\partial w}{\partial t} \|_{L^2(0,T;H^4)}] + k^2 \| \frac{\partial^3 w}{\partial t^3} \|_{L^2(0,T;H^0)} \} .$$

We remark that sufficient conditions for w to have the regularity required by lemma 3 are that

a) $\qquad F_{tt} \in L^2(0,T;H^0)$

(4.3) b) $\qquad F(x,0) = 0$ for $x = 0$ and $x = 1$, and

c) $\qquad \frac{\partial}{\partial x} (p \frac{\partial F(x,0)}{\partial x} - qF(x,0) + F_t(x,0) = 0 ,$ for $x = 0,1,$

where F is given in (2.2). These are natural compatibility assumptions on the data whose satisfactions are assumed with the hypothesis of the existence of a sufficiently smooth classical solution of the problem (1.1). When (4.3) is satisfied, we have by standard a priori estimates that for $n = 0,\ldots,N_T$,

$$(4.4) \qquad |w(\cdot,nk) - W_n|_\infty \leq C_5 \{ h_1^4 + k^2 \}$$

where C_5 depends only upon the data f_1, f_2, p, and q. Using the inverse properties (3.4) satisfied by the subspace $S_{h_1}^4$, we easily obtain the following result by a standard technique.

<u>Corollary (3.1).</u> There exists a constant C_6 such that for $n = 0,\ldots,N_T$,

$$(4.5) \qquad |w(\cdot,nk) - W_n|_{1,\infty} \leq \frac{C_6}{h_1} [h_1^4 + k^2] .$$

Our next lemma allows us to obtain L^∞-estimates for the error in approximation of (2.3) by (3.7) for times bounded away from $t = 0$ (i.e. $0 < t_0 \leq t$).

-10-

Lemma 4: (c.f. Baker, et. al [1]). Let z^ψ be the solution of (2.3) and $\{z_n^\psi\}$ the approximation given by (3.7). Then there exists constants $C_7 = C_7(t_0)$ such that for $n = N_0, \ldots, N_T$,

$$(4.6) \qquad |z^\psi(\cdot, nk) - z_n^\psi|_\infty \leq C_7\{h^2 + k^2\} \|\psi\|_0.$$

Again, using our inverse assumptions on S_h^2, we obtain

Corollary (4.1). There exists a constant C_8 such that for $n = N_0, \ldots, N_T$,

$$(4.7) \qquad |z^\psi(\cdot, nk) - z_n^\psi|_{1,\infty} \leq \frac{C_8}{h}[h^2 + k^2]\|\psi\|_0.$$

We shall also require a special stability result for the approximation scheme (3.6). Since the proof of the following lemma is quite technical, we shall defer it until after the proof of our main result.

Lemma 5. Let $\{W_n\}$ be the solution of (3.6). If for some constant $\tau_0 > 0$, we restrict k and h_1 such that $kh_1^2 \leq \tau_0$, then there is a constant C_9 such that

$$(4.8) \qquad |W_n|_{1,\infty} \leq C_9 \|F\|_{L^\infty(0,T;H^0)}.$$

The last lemma which we shall state gives an a priori estimate for the linear programming problem defined in (3.15). Again the proof of this lemma will be deferred until after the proof of our main result.

Lemma 6. Let $\varphi_h \in K_h$ be the solution of the linear programming problem (3.15). Then there exists a constant C_{10} such that

$$(4.9) \qquad \max_{n=N_0,\ldots,N_T} \left| g^*(nk) - p(0)[f_2^*(nk) - f_1^*(nk)] - p(0)\frac{\partial}{\partial x} W_n^*(0) - p(0)\frac{\partial}{\partial x} Z_n^{\varphi_h - r^*}(0) \right| = J(\varphi_h)$$

$$\leq C_{10}\{\varepsilon_0 + h + [h_1^4 + k^2]/h_1 + [h^2 + k^2]/h\}.$$

We are now in a position to state our major result and prove it using Lemmas 1-6.

Theorem 1: Let u be the solution of Problem (P) and $\{U_n\}$ be the approximation defined by Problem (P_A) and (3.11). Suppose that hypotheses (H1)-(H6) are satisfied, that F (defined by (2.2)) satisfies the regularity conditons (4.3) and that $\varphi = u(x,0)$ satisfies $\|\varphi\|_1 \leq C_{11}$ for some constant $C_{11} > 0$. If the mesh sizes k, h, and h_1 are chosen to satisfy $k = h = h_1^2$, then there exists a constant C_{12} which is independent of k such that for $n = N_0, \ldots, N_T$,

$$(4.10) \qquad |u(\cdot, nk) - U_n|_{1,\infty} \leq C_{12}[\varepsilon_0 + k]^\gamma$$

-11-

where ε_0 and γ are the constants defined by (H5) and (2.7) respectively.

Proof: In the reformulation of problem (P), we wrote the solution as

$$(4.11) \qquad u = u^{\varphi - r} = w + z^{\varphi - r} + (1-x)f_1(t) + xf_2(t) \ .$$

From (3.11) we have

$$(4.12) \qquad U_n = W_n^* + Z_n^{\varphi_h - r*} + (1-x)f_1^*(nk) + xf_2^*(nk).$$

Using the triangle inequality, we obtain

$$\left| u(\cdot,nk) - U_n \right|_{1,\infty} \le \left| w(\cdot,nk) - W_n \right|_{1,\infty} + \left| W_n - W_n^* \right|_{1,\infty}$$

$$(4.13) \qquad\qquad + \left| z^{\varphi - r}(\cdot,nk) - Z_n^{\varphi_h - r*} \right|_{1,\infty} + 2\left| f_1(nk) - f_1^*(nk) \right| + 2\left| f_2(nk) - f_2^*(nk) \right|.$$

The first term on the right of (4.13) is bounded using (4.5) as follows

$$(4.14) \qquad \left| w(\cdot,nk) - W_n \right|_{1,\infty} \le C_6[h_1^4 + k^2]/h_1 \ .$$

From (H5), we see that

$$(4.15) \qquad \left| f_1(nk) - f_1^*(nk) \right| + \left| f_2(nk) - f_2^*(nk) \right| \le 2\varepsilon_0 \ .$$

Since $k = h_1^2$ by hypothesis, we can use Lemma 5, (2.2) and (H5) to bound the second term on the right side of (4.13). We obtain

$$\left| W_n - W_n^* \right|_{1,\infty} \le C_9 \| F - F^* \|_{L^\infty(0,T;H^0)}$$

$$(4.16) \qquad\qquad \le C\varepsilon_0 \ .$$

In order to treat the third term on the right of (4.13) we use the triangle inequality,

$$\left| z^{\varphi - r}(\cdot,nk) - Z_n^{\varphi_h - r*} \right|_{1,\infty}$$

$$(4.17) \qquad\qquad \le \left| z^{\varphi - r}(\cdot,nk) - z^{\varphi_h - r*}(\cdot,nk) \right|_{1,\infty} + \left| z^{\varphi_h - r*}(\cdot,nk) - Z_n^{\varphi_h - r*} \right|_{1,\infty}.$$

Using (3.3.b), (3.10.a), and (4.7) we have for $n = N_0, \ldots, N_T$,

$$\left| z^{\varphi_h - r*}(\cdot,nk) - Z_n^{\varphi_h - r*} \right|_{1,\infty} \le \frac{C_8}{h}[h^2 + k^2]\|\varphi_h - r*\|_0$$

$$(4.18) \qquad\qquad \le \frac{C_8}{h}[h^2 + k^2]\{|\varphi_h|_\infty + |r*|_\infty\}$$

$$\le C[h^2 + k^2]/h \ .$$

Combining the above estimates, we obtain

$$\left| u(\cdot,nk) - U_n \right|_{1,\infty} \le C\{\varepsilon_0 + [h_1^4 + k^2]/h_1 + [h^2 + k^2]/h\}$$

$$(4.19) \qquad\qquad + \left| z^{\varphi - r}(\cdot,nk) - z^{\varphi_h - r*}(\cdot,nk) \right|_{1,\infty} \ .$$

In order to estimate the last term on the right of (4.19), we shall use the continuous dependence result from (2.7). We obtain

$$(4.20) \quad \left| z^{\varphi-r}(\cdot,nk) - z^{\varphi_h-r^*}(\cdot,nk) \right|_{1,\infty} \le C_1 \| z_x^{\varphi-r}(0,\cdot) - z_x^{\varphi_h-r^*}(0,\cdot) \|_{[t_0,T]}^{\gamma} \ .$$

For any function $y(t) \in W^{2,\infty}[t_0,T]$, we now define $I_k y$ to be the piecewise linear interpolate of y on the time mesh of width k. Then we obtain (using properties of a linear interpolate)

$$\| z_x^{\varphi-r}(0,\cdot) - z_x^{\varphi_h-r^*}(0,\cdot) \|_{[t_0,T]}$$

$$(4.21) \qquad \le \| z_x^{\varphi-r}(0,\cdot) - I_k z_x^{\varphi-r}(0,\cdot) \|_{[t_0,T]} + \| I_k z_x^{\varphi_h-r^*}(0,\cdot) - z_x^{\varphi_h-r^*}(0,\cdot) \|_{[t_0,T]}$$

$$+ \max_{n=N_0,\ldots,N_T} \left| z_x^{\varphi-r}(0,nk) - z_x^{\varphi_h-r^*}(0,nk) \right| \ .$$

By standard results in approximation theory, the first two terms on the right of (4.21) are bounded by

$$(4.22) \qquad C \, k^2 \, [\| z_{xtt}^{\varphi-r}(0,\cdot) \|_{[t_0,T]} + \| z_{xtt}^{\varphi_h-r^*}(0,\cdot) \|_{[t_0,T]}] \ .$$

Next, differentiating equation (2.3.a) and using Lemma 2 and the Sobolev lemma, we obtain for any $\psi \in H^0$,

$$\| z_{xtt}^{\psi}(0,\cdot) \|_{[0,T]} \le C \| z^{\psi}(\cdot,t) \|_6$$

$$(4.23)$$

$$\le C \| \psi \|_0 \ , \qquad \text{for} \qquad 0 < t_0 \le t \le T \ .$$

Combining the above estimates and using (3.3) and (3.10.a) to see that $\| \varphi - r \|_0 \le 2M$ and $\| \varphi_h - r^* \|_0 \le 2M$, we get

$$\| z_x^{\varphi-r}(0,\cdot) - z_x^{\varphi_h-r^*}(0,\cdot) \|_{[t_0,T]}$$

$$(4.24) \qquad \le C \, k^2 + \max_{n=N_0,\ldots,N_T} \left| z_x^{\varphi-r}(0,nk) - z_x^{\varphi_h-r^*}(0,nk) \right|$$

$$\le C \, k^2 + \max_{n=N_0,\ldots,N_T} \left| \frac{\partial}{\partial x} z^{\varphi-r}(0,nk) - \frac{\partial}{\partial x} z_n^{\varphi_h-r^*}(0) \right|$$

$$+ \max_{n=N_0,\ldots,N_T} \left| \frac{\partial}{\partial x} z_n^{\varphi_h-r^*}(0) - \frac{\partial}{\partial x} z^{\varphi_h-r^*}(0,nk) \right| \ .$$

Using (3.3), (3.10), and (4.7) we estimate the last term on the right of (4.24) by

$$\max_{n=N_0,\ldots,N_T} \left| \frac{\partial}{\partial x} z_n^{\varphi_h - r^*}(0) - \frac{\partial}{\partial x} z^{\varphi_h - r^*}(0,nk) \right| \le \frac{C_8}{h} [h^2 + k^2] \| \varphi_h - r^* \|_0$$

(4.25)

$$\le C[h^2 + k^2]/h .$$

Next, using (2.4) and the triangle inequality, we see that the second term on the right of (4.24) is bounded by

$$\max_{n=N_0,\ldots,N_T} \left| \frac{\partial}{\partial x} z^{\varphi - r}(0,nk) - \frac{\partial}{\partial x} z_n^{\varphi_h - r^*}(0) \right|$$

$$\le \max_{n=N_0,\ldots,N_T} \left| \frac{g(nk)}{p(0)} - \frac{\partial w}{\partial x}(0,nk) - [f_2(nk) - f_1(nk)] - \frac{\partial}{\partial x} z_n^{\varphi_h - r^*}(0) \right|$$

$$\le \max_{n=N_0,\ldots,N_T} \left\{ \frac{1}{p(0)} |g(nk) - g^*(nk)| + \left| \frac{\partial w}{\partial x}(0,nk) - \frac{\partial}{\partial x} W_n(0) \right| \right.$$

$$\left. + \left| \frac{\partial}{\partial x} W_n(0) - \frac{\partial}{\partial x} W_n^*(0) \right| + |f_2(nk) - f_2^*(nk)| + |f_1(nk) - f_1^*(nk)| \right\}$$

$$+ \max_{n=N_0,\ldots,N_T} \frac{1}{p(0)} \left| g^*(nk) - p(0) \frac{\partial}{\partial x} W_n^*(0) - p(0) [f_2^*(nk) - f_1^*(nk)] \right.$$

$$\left. - p(0) \frac{\partial}{\partial x} z_n^{\varphi_h - r^*}(0) \right|$$

$$\equiv term_1 + term_2 .$$

Using (H5), (4.5), and (4.16) we can bound the first term on the right of (4.26). We obtain

(4.27) $$term_1 \le C\{\varepsilon_0 + [h_1^4 + k^2]/h_1\}.$$

We note from (3.9) and (3.10) that the second term on the right of (4.26) is just $J(\varphi_h)$ from our Problem (P_A). We then use Lemma 6 to obtain the a priori estimate

(4.28)
$$term_2 = J(\varphi_h)/p(0)$$
$$\le C_{10}\{\varepsilon_0 + h + [h_1^4 + k^2]/h_1 + [h^2 + k^2]/h \}.$$

Then combining (4.19), (4.20), and (4.24)-(4.28), we see that for $n = N_0,\ldots,N_T$

(4.29) $$\left| u(\cdot,nk) - U_n \right|_{1,\infty} \le C\{\varepsilon_0 + h + [h_1^4 + k^2]/h_1 + [h^2 + k^2]/h\}^\gamma .$$

-14-

Then choosing $h = k$ and $h_1 = k^{1/2}$, we obtain

$$(4.30) \qquad \left| u(\cdot, nk) - U_n \right|_{1,\infty} \leq C\{\varepsilon_0 + k\}^{\gamma}$$

for $n = N_0, \ldots, N_T$, which was to be proved. $\qquad \qquad /\!/$

Next, assume that the linear programming problem described in Section 3 is solved to within the tolerance

$$J(\varphi_h) \leq \sigma_1$$

for some $\sigma_1 > 0$. Replacing the estimate (4.28) by the above inequality, we obtain the following error estimate.

<u>Corollary T1</u>: Assume the hypotheses of Theorem 1 are satisfied. If $\varphi_h^* = \sum\limits_{i=1}^{H-1} C_i^* \, \Phi_i$, the solution of the linear programming problem defined by (3.15), satisfies $J(\varphi_h^*) \leq \sigma_1$, then for $n = N_0, \ldots, N_T$ we have for some constant $C_{13} > 0$,

$$\left| u(\cdot, nk) - U_n \right|_{1,\infty} \leq C_{13} [\varepsilon_0 + k + \sigma_1]^{\gamma}.$$

We shall finally give proofs for Lemmas 5 and 6 which were stated previously.

<u>Proof of Lemma 5</u>: To prove this lemma we will need to make use of results from elliptic regularity theory, spectral theory in Hilbert spaces, and the theory of interpolation spaces. We shall assume the reader is familiar with these concepts, since to provide detailed explanations would unduly lengthen the proof. In order to simplify the exposition, we first introduce some additional notation. Let Q be the solution operator for the two point boundary value problem

$$\text{a)} \qquad -\frac{\partial}{\partial x} (p \, \frac{\partial y}{\partial x}) + qy = f , \qquad 0 < x < 1 ,$$

$$(4.31) \ \text{b)} \qquad y(0) = 0 ,$$

$$\text{c)} \qquad y(1) = 0 .$$

(i.e., $y = Qf$). Let Q_{h_1} be the solution operator for the Galerkin approximation of (4.31). Then $y_{h_1} = Q_{h_1} f$ is defined by

$$(4.32) \qquad a(y_{h_1}, v_{h_1}) = (f, v_{h_1}) , \qquad v_{h_1} \in S_{h_1}^4 .$$

Now, set $L_{h_1} = Q_{h_1}^{-1}$ (i.e., the inverse of Q_{h_1} on $S_{h_1}^4$). For normed spaces X and Y, let $\mathcal{L}(X,Y)$ denote the space of linear operators from X to Y and $\| \cdot \|_{\mathcal{L}(X,Y)}$ the (operator) norm in this space. We shall first establish that

$$(4.33) \qquad \left| w_n \right|_{1,\infty} \leq C \| L_{h_1}^{7/8} \, w_n \|_0 .$$

-15-

We note that

$$|W_n|_{1,\infty} = |Q_{h_1} L_{h_1} W_n|_{1,\infty}$$

$$\leq |(Q_{h_1} - Q) L_{h_1} W_n|_{1,\infty} + |Q L_{h_1} W_n|_{1,\infty}$$

(4.34)

$$\leq |Q_{h_1} L_{h_1} W_n - I_{h_1} Q L_{h_1} W_n|_{1,\infty} + |I_{h_1} Q L_{h_1} W_n - Q L_{h_1} W_n|_{1,\infty}$$

$$+ |Q L_{h_1} W_n|_{1,\infty}$$

where I_{h_1} is the interpolation operator mapping $H_0^1 \to S_{h_1}^4$. By the inverse properties of $S_{h_1}^4$ given in (3.4), we see that

$$|Q_{h_1} L_{h_1} W_n - I_{h_1} Q L_{h_1} W_n|_{1,\infty}$$

(4.35)

$$\leq C h_1^{-3/2} \|Q_{h_1} L_{h_1} W_n - I_{h_1} Q L_{h_1} W_n\|_0$$

$$\leq C h_1^{-3/2} \{\|Q_{h_1} L_{h_1} W_n - Q L_{h_1} W_n\|_0 + \|Q L_{h_1} W_n - I_{h_1} Q L_{h_1} W_n\|_0\}.$$

Using standard properties of the Galerkin approximation, we note that the first term on the right of (4.35) can be estimated as follows:

(4.36) $$h_1^{-3/2} \|(Q_{h_1} - Q) L_{h_1} W_n\|_0 \leq C h_1^{-3/2} h_1^{7/4} \|Q L_{h_1} W_n\|_{7/4} .$$

Also, using (3.5.a), we obtain

(4.37) $$h_1^{-3/2} \|Q L_{h_1} W_n - I_{h_1} Q L_{h_1} W_n\|_0 \leq C h_1^{-3/2} h_1^{7/4} \|Q L_{h_1} W_n\|_{7/4} .$$

Then, using (3.5.b) and the Sobolev lemma, we see that

$$|I_{h_1} Q L_{h_1} W_n - Q L_{h_1} W_n|_{1,\infty} \leq |I_{h_1} Q L_{h_1} W_n|_{1,\infty} + |Q L_{h_1} W_n|_{1,\infty}$$

(4.38)

$$\leq (C_3 + 1) |Q L_{h_1} W_n|_{1,\infty}$$

$$\leq C \|Q L_{h_1} W_n\|_{7/4} .$$

We then collect terms in the above inequalities and use a standard a priori estimate for elliptic problems (i.e. $Q \in \mathcal{L}(H^{-1/4}, H^{7/4})$ [10]) to obtain

$$|W_n|_{1,\infty} \leq C \|Q L_{h_1} W_n\|_{7/4}$$

(4.39)

$$\leq C \|L_{h_1} W_n\|_{-1/4} ,$$

where the last norm is defined below in (4.41). Then, in order to obtain (4.33) we need

-16-

only show that

$$(4.40) \qquad \|L_{h_1} w_n\|_{-1/4} \leq C \|L_{h_1}^{7/8} w_n\|_0 \; .$$

Let $P:L^2 \to S_{h_1}^4$ be the L^2 projection operator into $S_{h_1}^4$. Note that L_{h_1} is self-adjoint, so $L_{h_1}^{1/8}$ is also self-adjoint. Then we see that

$$\|L_{h_1} w_n\|_{-1/4} \equiv \sup_{v \in C_0^\infty} \frac{(L_{h_1} w_n, v)}{\|v\|_{1/4}}$$

$$(4.41) \qquad = \sup_{v \in C_0^\infty} \frac{(L_{h_1} w_n, Pv)}{\|v\|_{1/4}}$$

$$= \sup_{v \in C_0^\infty} \frac{(L_{h_1}^{7/8} w_n, L_{h_1}^{1/8} Pv)}{\|v\|_{1/4}}$$

$$\leq \|L_{h_1}^{7/8} w_n\|_0 \sup_{v \in C_0^\infty} \frac{\|L_{h_1}^{1/8} Pv\|_0}{\|v\|_{1/4}} \; .$$

Next, since we have that

$$(4.42) \qquad \|L_{h_1}^{1/2} Pv\|_0 = [a(Pv, Pv)]^{1/2} \leq C \|Pv\|_1$$

and

$$(4.43) \qquad \|L_{h_1}^0 Pv\|_0 \leq \|Pv\|_0 \; ,$$

we can use an interpolation theorem due to Heinz [9] (compare also with [10]) to establish that

$$(4.44) \qquad \|L_{h_1}^{1/8} Pv\|_0 \leq C \|Pv\|_{1/4} \; .$$

Then, since approximation properties of $S_{h_1}^4$ yield

$$(4.45) \qquad \|Pv\|_{1/4} \leq C \|v\|_{1/4} \; ,$$

we can combine (4.41), (4.44) and (4.45) to obtain

$$(4.46) \qquad \|L_{h_1} w_n\|_{-1/4} \leq C \|L_{h_1}^{7/8} w_n\|_0 \; .$$

Now, to establish (4.8) we solve a set of first order difference equations arising from (3.6) to obtain

$$(4.47) \qquad \begin{aligned} w_n = k \sum_{j=0}^{n-1} \Bigg[&\left\{ [I + \frac{k}{2} L_{h_1}]^{-1} [I - \frac{k}{2} L_{h_1}] \right\}^j \; \cdot \\ &\cdot [I + \frac{k}{2} L_{h_1}]^{-1} P F(\cdot, [n - \frac{1}{2} - j]k) \Bigg]. \end{aligned}$$

-17-

Then we see that

$$\|L_{h_1}^{7/8} w_n\|_0 \leq \|kL_{h_1}^{7/8}[I + \frac{k}{2} L_{h_1}]^{-1} PF(\cdot, [n - \frac{1}{2}]k)\|_0$$

$$(4.48) \qquad + k \sum_{j=1}^{n-1} \|L_{h_1}^{7/8} \{[I + \frac{k}{2} L_{h_1}]^{-1}[I - \frac{k}{2} L_{h_1}]\}^j [I + \frac{k}{2} L_{h_1}]^{-1}\|_{\mathcal{L}(H^0, H^0)} \cdot$$

$$\cdot \|PF(\cdot, [n - \frac{1}{2} - j]k\|_0 \ .$$

For the first term on the right of (4.48), we obtain

$$\|k L_{h_1}^{7/8} [I + \frac{k}{2} L_{h_1}]^{-1} PF(\cdot, [n - \frac{1}{2}]k)\|_0$$

$$\leq 2^{7/8} k^{1/8} \| (\frac{k}{2})^{7/8} L_{h_1}^{7/8} [I + \frac{k}{2} L_{h_1}]^{-1}\|_{\mathcal{L}(H^0, H^0)} \|PF(\cdot, [n - \frac{1}{2}]k)\|_0$$

$$(4.49)$$

$$\leq 2^{7/8} k^{1/8} \sup_{0 < \lambda < \infty} \left\{ \frac{\lambda^{7/8}}{1 + \lambda} \right\} \|PF(\cdot, [n - \frac{1}{2}]k)\|_0$$

$$\leq 2 \|PF(\cdot, [n - \frac{1}{2}]k)\|_0 \qquad (\text{for } k \leq 2).$$

Since it is known that the eigenvalues Λ_i of L_{h_1} satisfy $0 < \Lambda_i \leq \beta_0 h_1^{-2}$, the mesh ratio restriction $k h_1^{-2} \leq \tau_0$ contained in the statement of Lemma 5 implies that

$$(4.50) \qquad \Lambda_i \leq \beta_0 \tau_0 / k \ .$$

From [1], we know that there exist constants C_0 and K_0 such that for $j = 1, 2, \ldots$, and $0 < \lambda \leq \frac{\beta_0 \tau_0}{k}$,

$$(4.50) \qquad \left| \frac{1 - \frac{k}{2} \lambda}{1 + \frac{k}{2} \lambda} \right|^j \leq K_0 e^{-C_0 \frac{k}{2} \lambda j} \ .$$

Then, since clearly $e^{-Cx} x^{7/8}$ is bounded for all $x \geq 0$, we use (4.50) to see that

$$\|L_{h_1}^{7/8} \{[I + \frac{k}{2} L_{h_1}]^{-1}[I - \frac{k}{2} L_{h_1}]\}^j [I + \frac{k}{2} L_{h_1}]^{-1}\|_{\mathcal{L}(H^0, H^0)}$$

$$\leq \sup_{0 < \lambda < \frac{\beta_0 \tau_0}{k}} \left| \frac{1 - \frac{k}{2} \lambda}{1 + \frac{k}{2} \lambda} \right|^j \frac{\lambda^{7/8}}{1 + \frac{k}{2} \lambda}$$

$$(4.51) \qquad \leq \sup_{0 < \lambda < \frac{\beta_0 \tau_0}{k}} e^{-C_0 \frac{k}{2} \lambda j} \lambda^{7/8}$$

$$\leq \sup_{0 < \lambda < \frac{\beta_0 \tau_0}{k}} e^{-C_0 \frac{k}{2} \lambda j} \frac{(\frac{k}{2} \lambda j)^{7/8}}{(jk/2)^{7/8}}$$

$$\leq C_{12} (jk/2)^{-7/8} \ .$$

Thus combining the above estimates, we have

$$\| L_{h_1}^{7/8} W_n \|_0 \leq 2 \| PF(\cdot, [n - \frac{1}{2}]k) \|_0 + \| PF \|_{L^\infty(0,T;H^0)} \, k \sum_{j=1}^{n-1} \frac{C_{12}}{(jk/2)^{7/8}}$$

$$\leq \| PF \|_{L^\infty(0,T;H^0)} \, [2 + \frac{C_{12}}{2^{7/8}} \int_0^T \frac{1}{t^{7/8}} \, dt]$$

(4.52)

$$\leq \| PF \|_{L^\infty(0,T:H^0)} \, [2 + \frac{8C_{12}}{2^{7/8}} T^{1/8}]$$

$$\leq C \| F \|_{L^\infty(0,T:H^0)} \, .$$

Finally, combining (4.39), (4.46) and (4.52) proves Lemma 5 . //

<u>Proof of Lemma 6</u>: From (H5), (3.9), (3.10), (4.5) and (4.16) we see that for all $\psi_h \in K_h$

$$J(\varphi_h) \leq J(\psi_h)$$

$$= \max_{n=N_0,\ldots,N_T} |g^*(nk) - p(0)[f_2^*(nk) - f_1^*(nk)] - p(0)\frac{\partial}{\partial x} W_n^*(0) - p(0)\frac{\partial}{\partial x} Z_n^{\psi_h - r^*}(0)|$$

$$\leq \max_{n=N_0,\ldots,N_T} \{ |g^*(nk) - g(nk)| + p(0)|f_2^*(nk) - f_2(nk)| + p(0)|f_1^*(nk) - f_1(nk)|$$

$$+ p(0)|\frac{\partial}{\partial x} W_n^*(0) - \frac{\partial}{\partial x} W_n(0)| + p(0)|\frac{\partial}{\partial x} W_n(0) - \frac{\partial w}{\partial x}(0,nk)|$$

$$+ p(0)|\frac{\partial}{\partial x} Z_n^{\psi_h - r^*}(0) - \frac{\partial}{\partial x} z^{\varphi - r}(0,nk)| \}$$

$$\leq C[\varepsilon_0 + (h_1^4 + k^2)/h_1] + \max_{n = N_0,\ldots,N_T} p(0)|\frac{\partial}{\partial x} Z_n^{\psi_h - r^*}(0) - \frac{\partial}{\partial x} z^{\varphi - r}(0,nk)| \, .$$

Now

$$|\frac{\partial}{\partial x} Z_n^{\psi_h - r^*}(0) - \frac{\partial}{\partial x} z^{\varphi - r}(0,nk)|$$

(4.54)

$$\leq |\frac{\partial}{\partial x} Z_n^{\psi_h - r^*}(0) - \frac{\partial}{\partial x} z^{\psi_h - r^*}(0,nk)| + |\frac{\partial}{\partial x} z^{\psi_h - \varphi + r - r^*}(0,nk)|.$$

From (4.7), (3.3), and (3.10) we see that for $n = N_0,\ldots,N_T$,

$$|\frac{\partial}{\partial x} Z_n^{\psi_h - r^*}(0) - \frac{\partial}{\partial x} z^{\psi_h - r^*}(0,nk)| \leq \frac{C_8}{h} [h^2 + k^2] \| \psi_h - r^* \|_0$$

(4.55)

$$\leq C[h^2 + k^2]/h .$$

Next using the Sobolev lemma, Lemma 2, and (3.3.c), we obtain for $n = N_0, \ldots, N_T$, and for any $\psi_k \in K_h$,

$$\left| \frac{\partial}{\partial x} z^{\psi_h - \varphi + r - r^*}(0, nk) \right| \leq \| z^{\psi_h - \varphi + r - r^*} \|_2$$

$$(4.56) \qquad\qquad \leq C \| \psi_h - \varphi + r - r^* \|_0$$

$$\leq C[\varepsilon_0 + \| \psi_h - \varphi \|_0] .$$

Choose ψ_h such that

$$(4.57) \qquad
\begin{array}{ll}
\text{a)} & \psi_h(ih) = \varphi(ih) , \quad i = 1, \ldots, H-1, \\
\text{b)} & \psi_h(0) = f_1^*(0) , \\
\text{c)} & \psi_h(1) = f_2^*(0) .
\end{array}
$$

Then $\psi_h \in K_h$ and ψ_h is almost equal to $I_h \varphi$, the piecewise linear interpolate of φ. Then from (3.5),

$$(4.58) \qquad \| \psi_h - \varphi \|_0 \leq \| \psi_h - I_h \varphi \|_0 + \| I_h \varphi - \varphi \|_0$$

$$\leq \| \psi_h - I_h \varphi \|_0 + Ch \| \varphi \|_1 .$$

But we see that

$$(4.59) \qquad \psi_h - I_h \varphi = \begin{cases} 0 & , h \leq x \leq (H-1)h , \\[2mm] (1 - \dfrac{x}{h})[f_1^*(0) - f_1(0)] & , 0 \leq x \leq h , \\[2mm] [\dfrac{x}{h} - (H-1)][f_2^*(0) - f_2(0)] & , (H-1)h \leq x \leq 1 , \end{cases}$$

so we have

$$(4.60) \qquad \| \psi_h - I_h \varphi \|_0 \leq C \varepsilon_0 .$$

Combining (4.53)-(4.60), we obtain

$$(4.61) \qquad J(\varphi_h) \leq C[\varepsilon_0 + h + (h_1^4 + k^2)/h_1 + (h^2 + k^2)/h] . \qquad\qquad /\!/$$

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A procedure for the numerical approximation of the Cauchy problem for the following linear parabolic partial differential equation is defined: $u_t - (p(x)u_x)_x + q(x)u=0$, $0<x<1$, $0<t<T$; $u(0,t)=f_1(t)$, $0<t\leq T$; $u(1,t)=f_2(t)$, $0<t\leq T$; $p(0)u_x(0,t)=g(t)$, $0<t_0\leq t\leq T$; $|u(x,t)|\leq M$, $0<x<1$, $0\leq t<T$. The procedure involves Galerkin type numerical methods for related parabolic initial boundary-value problems and a linear programming problem. Explicit a priori error estimates are presented for the entire discrete procedure when the data f_1, f_2, and g are known only approximately.

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9